The ring of integers of a function field and its primes

Holly Green

University College London

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Goal and motivation

Let $q = p^r$. A function field is

- a finitely generated field $K/\mathbb{F}_q$ of transcendence degree 1
- $\mathbb{F}_q(C)$ for a smooth projective curve $C/\mathbb{F}_q$: in particular if $C : F(x, y) = 0$ this is the fraction field of

$$\mathbb{F}_q[x, y]/(F(x,$$

Valuations on $\mathbb{F}_q(x)$

$$\mathbb{Q} \quad \leftarrow \quad \mathbb{Z} = \square_{p \text{ prime}} \{x \quad \mathbb{Q} : |x|_p \leq 1\}$$

$$\mathbb{F}_q(x) = \mathbb{F}_q(\mathbb{P}^1) \quad \leftarrow \quad \mathbb{F}_q[x]$$

Question

Is $\mathbb{F}_q[x]$ cut out by valuation bounds in the same way as $\mathbb{Z}$?

Let $f \quad \mathbb{F}_q(x)$ be a rational function on $\mathbb{P}^1$. Fix $P \quad \mathbb{P}^1(\mathbb{F}_{q^n})$ for some $n \geq 1$, define

$$\mathrm{ord}_P(f$$

Absolute values on $\mathbb{F}_q(x)$

Let $f \quad \mathbb{F}_q(x) = \mathbb{F}_q(\mathbb{P}^1)$ and fix $P \quad \mathbb{P}^1(\mathbb{F}_{q^n})$ for some $n \geq 1$.

Definition

The *absolute value of* f *at* P is $|f|_P = (q^n)^{-\mathrm{ord}_P(f)}$.

For example, if $f = x/(x^2 - 2)$ and $q = 5$ then

$$|f|_0 = 5^{-1}, \qquad |f|_{\pm\sqrt{2}} = 25^1, \qquad |f$$

The analogue of $\mathbb{Z} \hookrightarrow \mathbb{Q}$ for $\mathbb{F}_q($

Properties of $O_{K,S}$

More generally, for *smooth* $C : F(x, y) = 0$ over $\mathbb{F}_q$, taking $S = \{\text{points at } \infty\}$ gives

$$O$$

The primes of $O_{K,S}$

Recall that for $K = \mathbb{F}_q(C$

The primes of $O_{K,S}$

Definition

For a closed point $P \;/\; S$, the *prime ideal of* $O_{K,S}$ *at* P is

$$p_{P,S} = \{f \quad K : f \text{ has a zero at } P \text{ and no poles outside of } S\}.$$

Proposition

Every prime ideal of $O_{K,S}$ is of the form $p_{P,S}$ for a closed point $P \;/\; S$. There's a correspondence between the primes of $O_{K,S}$ and the Galois orbits of points in $C(\overline{\mathbb{F}}_q)$ not in S.

When

Example

Let $C : y^2 = x^3 - x$, $K = \mathbb{F}_q(C)$ and $S = \{\infty\}$. We saw previously that

The Chinese Remainder Theorem

Fix $K = \mathbb{F}_q(C)$, S a finite set of closed points. We have a ring $O_{K,S}$ with prime ideals $\mathfrak{p}_{P,S}$.

The Chinese Remainder Theorem

Let $P, Q \ / \ S$ be distinct closed points. There's an isomorphism

$$O_{K,S}/(\mathfrak{p}_{P,S} \cap \mathfrak{p}_{Q,S}) \longrightarrow O_{K,S}/\mathfrak{p}_{P,S} \times O_{K,S}/\mathfrak{p}_{Q,S}.$$

In particular, given $s, t \quad \overline{\mathbb{F}}_q$ defined over the residue fields of P and Q respectively, there's some $f \quad O_{K,S}$ such that $f(P) = s$ and

Thank you for listening!

Any questions?